Math 155, ***Lecture Notes- Bonds*** **Name____________**

Section 10.3 *Parametric Equations and Calculus*

Now that we can graph parametric equations, we can consider extending the concepts of continuity and differentiation to these curves. How do we find equations of tangent lines? How do we take higher order derivatives? How do we find concavity? What will Arc Length look like with parametric equations?

Definition of a Smooth Curve

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called **smooth** if f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called **piecewise smooth** if it is smooth on each subinterval of some partition of I.

THEOREM 10.7 Parametric Form of the Derivative

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

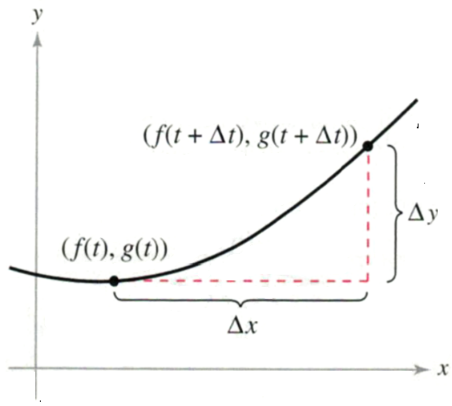
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

For higher order derivatives, use

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} \qquad \text{SECOND DERIVATIVE}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{dx/dt} \qquad \text{THIRD DERIVATIVE}$$

Notice that the denominator for each higher-order derivative is **always** *dx*/*dt*.


y
$(f(t+\Delta t), g(t+\Delta t))$
$(f(t), g(t))$
Δy
Δx
x

Ex. 1: Find $\dfrac{dy}{dx}$ for $\begin{cases} x(t) = \sqrt[3]{t} \\ y(t) = 4 - t \end{cases}$

Ex. 2: Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for $\begin{cases} x(\theta) = \cos(\theta) \\ y(\theta) = 3\sin(\theta) \end{cases}$ at $\theta = 0$.

More Ex. 2:

Ex. 3: Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for $\begin{cases} x(\theta) = \cos(\theta) \\ y(\theta) = 3\sin(\theta) \end{cases}$ at $\theta = \dfrac{\pi}{4}$.

More Ex. 3:

Ex. 4: Find the equation of the tangent line to the curve, C, defined by the equation $\frac{x^2}{9}+\frac{y^2}{16}=1$ at the point M, $\left(\frac{-3\sqrt{2}}{2},y\right)$, $y>0$.

More Ex. 4:

More Ex. 4:

Ex. 5: Find the equations of the tangent line at the point where the curve crosses itself.

$$\begin{cases} x(t) = t^3 - 6t \\ y(t) = t^2 \end{cases}$$

More Ex. 5:

More Ex. 5:

If $\frac{dy}{dt}=0$ and $\frac{dx}{dt}\neq 0$ when $t=t_0$, then the parametric curve represented by $x=f(t)$ and $y=g(t)$ has a **horizontal tangent** at $\left(f(t_0),g(t_0)\right)$.

If $\frac{dx}{dt}=0$ and $\frac{dy}{dt}\neq 0$ when $t=t_0$, then the parametric curve represented by $x=f(t)$ and $y=g(t)$ has a **vertical tangent** at $\left(f(t_0),g(t_0)\right)$.

If $\frac{dx}{dt}=0$ **and** $\frac{dy}{dt}=0$ when $t=t_0$, then $\frac{dy}{dx}=\frac{0}{0}$ yields and indeterminate form. We need to study this situation on a case-by case-basis and we must consider the graph behavior near this point on the curve, since the indeterminate form cannot tell us what is happening.

Ex. 6: Find the points of horizontal tangency and vertical tangency.

$$\begin{cases} x(t)=2t \\ y(t)=2\left[1-\cos(t)\right] \end{cases} \quad \text{for } -1\leq t\leq 2\pi$$

Ex. 7: Determine the t-intervals on which the curve is concave downward, or concave upward.

$$\begin{cases} x(t) = 2t + \ln(t) \\ y(t) = 2t - \ln(t) \end{cases}$$

More Ex. 7:

THEOREM 10.8 Arc Length in Parametric Form

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Ex. 8: Write an integral that represents the arc length of the curve over $1 \leq t \leq 6$.

$$\begin{cases} x(t) = \ln(t) \\ y(t) = t+1 \end{cases}$$

Ex. 9: Find the circumference of a circle with radius a.

$$\begin{cases} x(t) = a\cos(t) \\ y(t) = a\sin(t) \end{cases} \quad \text{for } 0 \le t \le 2\pi$$